Recall:

For $f = F'$ we have

$$F(x)\Big|_a^b = F(b) - F(a) = \int_a^b f(t)\,dt$$

We also write

$$\int f(x)\,dx = F(x) \; .$$

Example 7.4:

i) Let $s \in \mathbb{R}$, $s \neq -1$. Then we have

$$\int_a^b x^s\,dx = \frac{x^{s+1}}{s+1}\Big|_a^b$$

where we have the following restrictions for the integration interval:

for $s \in \mathbb{N}$, $a, b \in \mathbb{R}$ arbitrary; for $s \leq -2$, the interval $[a, b]$ should not contain 0 (otherwise F not continuous there)

ii) For $a, b > 0$ we have

$$\int_a^b \frac{dx}{x} = \log x \Big|_a^b \; .$$

For $a, b < 0$ we have

$$\int_a^b \frac{dx}{x} = \log(-x)\Big|_a^b, \quad \text{as } \frac{d}{dx}\log(-x) = \frac{1}{x}$$

for $x < 0$.

Summarizing, we get:

$$\int \frac{dx}{x} = \log|x|, \text{ for } x \neq 0.$$

(0 should not lie in the integration interval)

iii) $\int \sin x \, dx = -\cos x$

iv) $\int \cos x \, dx = \sin x$

v) $\int \exp x \, dx = \exp x$

vi) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$ for $|x| < 1$.

vii) $\int \frac{dx}{1+x^2} = \arctan x$.

viii) $\int \frac{dx}{\cos^2 x} = \tan x$

where $\cos x \neq 0$ in the integration interval.

Proposition 7.11 (Substitution rule):

Let $f : I \longrightarrow \mathbb{R}$ be a continuous function and $\varphi : [a,b] \longrightarrow \mathbb{R}$ a continuously differentiable function with $\varphi([a,b]) \subset I$. Then we have

$$\int_a^b f(\varphi(t))\,\varphi'(t)\,dt = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx .$$

Proof:

Let $F : I \longrightarrow \mathbb{R}$ be an indefinite integral of f. For the function $F \circ \varphi : [a,b] \longrightarrow \mathbb{R}$ we have using the chain rule

$$(F \circ \varphi)'(t) = F'(\varphi(t))\,\varphi'(t) = f(\varphi(t))\,\varphi'(t)$$

$$\overset{\text{Th. 7.2}}{\Longrightarrow} \int_a^b f(\varphi(t))\,\varphi'(t)\,dt = (F \circ \varphi)(t)\Big|_a^b$$

$$= F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx$$

Notation:

$$d\varphi(t) := \varphi'(t)\,dt$$

and $\displaystyle\int_a^b f(\varphi(t))\,d\varphi(t) = \int_{\varphi(a)}^{\varphi(b)} f(x)\,dx .$

Example 7.5:

i) $$\int_a^b f(t+c)\,dt = \int_{a+c}^{b+c} f(x)\,dx, \quad (\text{substitution } \varphi(t) = t+c)$$

ii) For $c \neq 0$ we have
$$\int_a^b f(ct)\,dt = \frac{1}{c}\int_{ac}^{bc} f(x)\,dx, \quad (\varphi(t) = ct)$$

iii) $$\int_a^b t f(t^2)\,dt = \frac{1}{2}\int_{a^2}^{b^2} f(x)\,dx, \quad (\varphi(t) = t^2)$$

iv) Let $\varphi : [a,b] \longrightarrow \mathbb{R}$ be a continuously differentiable function with $\varphi(t) \neq 0$ for all $t \in [a,b]$. Then we have
$$\int_a^b \frac{\varphi'(t)}{\varphi(t)}\,dt = \log|\varphi(t)|\Big|_a^b, \quad \left(f(x) = \frac{1}{x},\ x = \varphi(t)\right)$$

v) Let $[a,b] \subset (-\frac{\pi}{2}, \frac{\pi}{2})$. Then we have
$$\int_a^b \tan t\,dt = \int_a^b \frac{\sin t}{\cos t}\,dt = -\log\cos t\Big|_a^b .$$

vi) In order to compute $\int_a^b \frac{dx}{1-x^2}$, where $-1, 1 \notin [a,b]$, one uses the following:

As $1-x^2 = (1-x)(1+x)$, we solve for $\alpha, \beta \in \mathbb{R}$ such that

$$\frac{1}{1-x^2} = \frac{\alpha}{1-x} + \frac{\beta}{1+x},$$

or

$$\frac{1}{1-x^2} = \frac{(\alpha+\beta)+(\alpha-\beta)x}{1-x^2}$$

$\Rightarrow \alpha = \beta = \frac{1}{2}$. This implies

$$\int_a^b \frac{dx}{1-x^2} = \frac{1}{2}\left(\int_a^b \frac{dx}{1-x} + \int_a^b \frac{dx}{1+x}\right)$$

$$= \frac{1}{2}\left(\int_a^b \frac{dx}{1+x} - \int_a^b \frac{dx}{x-1}\right)$$

$$= \frac{1}{2}\left(\log|x+1| - \log|x-1|\right)\Big|_a^b$$

$$= \frac{1}{2}\log\left|\frac{x+1}{x-1}\right|\Big|_a^b$$

vii) In order to compute $\int \frac{dx}{\sqrt{1+x^2}}$, use the substitution $x = \sinh t = \frac{1}{2}(e^t - e^{-t})$. As

$d\sinh t = \cosh t \, dt$, $\cosh^2 t - \sinh^2 t = 1$,

$$\operatorname{arsinh} x = \log\left(x + \sqrt{1+x^2}\right), \quad \text{(Homework)}$$

we get with $u := \operatorname{Arsinh} a$, $v := \operatorname{Arsinh} b$

$$\int_a^b \frac{dx}{\sqrt{1+x^2}} = \int_u^v \frac{d\sinh t}{\sqrt{1+\sinh^2 t}} = \int_u^v \frac{\cosh t}{\cosh t}\, dt = t\Big|_u^v$$

$$= \log\left(x + \sqrt{1+x^2}\right)\Big|_a^b .$$

viii) Computation of $\displaystyle\int_a^b \frac{dx}{\sqrt{x^2-1}}$, $(a,b>1)$.

We substitute $x = \cosh t = \frac{1}{2}(e^t + e^{-t})$. As

$$d\cosh t = \sinh t\, dt$$

$$\operatorname{Ar}\cosh x = \log\left(x + \sqrt{x^2-1}\right),$$

it follows with $u := \operatorname{Arcosh} a$, $v := \operatorname{Arcosh} b$

$$\int_a^b \frac{dx}{\sqrt{x^2-1}} = \int_u^v \frac{\sinh t}{\sinh t}\, dt = t\Big|_u^v$$

$$= \log\left(x + \sqrt{x^2-1}\right)\Big|_a^b .$$

Proposition 7.12 (Partial integration):

Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be two continuously differentiable functions. Then

$$\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x) \Big|_a^b - \int_a^b g(x) f'(x)\, dx$$

A short hand notation for this formula is:

$$\int f\, dg = fg - \int g\, df.$$

Proof:

For $F := fg$ we have according to the product law:

$$F'(x) = f'(x) g(x) + f(x) g'(x),$$

and thus according to Theorem 7.2:

$$\int_a^b f'(x) g(x)\, dx + \int_a^b f(x) g'(x)\, dx = F(x) \Big|_a^b$$

$$= f(x)\, g(x) \Big|_a^b$$

$\Rightarrow$ from this the claim follows.

$\square$

Example 7.6:

i) Let $a, b > 0$. In order to compute $\int_a^b \log x \, dx$ we set $f(x) = \log x$, $g(x) = x$.

$$\int_a^b \log x \, dx = x \log x \Big|_a^b - \int_a^b x \, d\log x$$

$$= x \log x \Big|_a^b - \int_a^b dx$$

$$= x(\log x - 1)\Big|_a^b .$$

ii) Computation of $\int \arctan x \, dx$.

$$\int \arctan x \, dx = x \arctan x - \int x \, d\arctan x$$

As $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$, we get

$$\int x \, d\arctan x = \int \frac{x}{1+x^2} \, dx = \left(\begin{array}{c}\text{substitution} \\ t = x^2\end{array}\right)$$

$$= \frac{1}{2} \int \frac{dt}{1+t} = \frac{1}{2} \log(1+t)$$

$$= \frac{1}{2} \log(1+x^2)$$

$$\Longrightarrow \int \arctan x \, dx = x \arctan x - \frac{1}{2} \log(1+x^2)$$